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Numerical Solution of the k=3 Queens problem!

A RECREATIVE COMBINATORIAL ANALYSIS

(based on work by Antal Pinter at 1973)

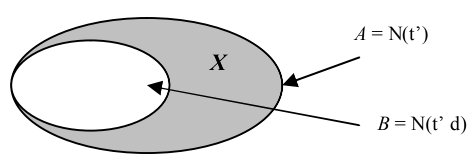
INTRODUCTION

One of the most popular version of theoretical chess problems in the recreative mathematics is the eight Queens problem, that is, in how many ways can eight Queens of the chessboard arranged in such a way that they do not attack each other. The task is originate still from 1848, given by a German chess player, *M. Betzel*, and then dealt with it *dr. F. Nauck* and itself *Gauss* too. The task was solved, although that is particularly not so difficult, anyone could find all the positions of the various 92 with a small effort, but the problem in his general form on the *n*x*n* board is still unsolved! We can informed about various records from time to time, recently computer tournaments held annually considering this matter, but according to my knowledge, the maximum table size where solution exists is still only up to $n = 26$.
The main interest in this subject is not ends with finding out of each possible position, in fact, it only begins with this. The ultimate goal is the exact **mathematical expression** of the problem of arbitrary number of *k*-Queen's on the *n*x*n* -sized table, which has also remain unsolved! Mathematical formula is known only up to $k = 6$ Queens, and in this work I present step-by-step the solution and the path leading to for $k = 3$ Queens!
In contrast of *E.Landau*'s formula from 1896, given separately for odd and even tables, here is shown a detailed procedure for obtaining a generalized unique formula!

1. Setting up the conceptual bases

Solving the problem we based on the idea that we must subtract solutions of the rook problem, **N*(t')*** with all those positions where the queens attacks each other *only diagonally!* Thus the basic relation symbolically [1)]

$$\boldsymbol{N(t'd') = N(t') - N(t'd)}$$



$$B, X \subset A \qquad B \cap X = \varnothing \qquad X = A \setminus B \tag{1}$$

Value of ***N*(*t'*)** is known, equal with the product of the $C(n,k)$ combination and the $V(n,k)$ variation, or

$$\boldsymbol{N(t') = \binom{n}{k}^2 k!} \tag{2}$$

So, our task here is to determine the value of ***N*(*t'd*)** , which were be reached by enumerating all those cases where *either two queens* attacks each other diagonally! With the a_1, a_2, a_3 queens this is possible in the following manner:

$\sigma = N(a_1\text{-}a_2)$
$\lambda = N(a_1\text{-}a_3)$
$\delta = N(a_2\text{-}a_3)$ that is, $\boldsymbol{B = N(t'd) = \sigma \cup \lambda \cup \delta}$ (3)

[1)] N- stands for enumerating function, t- for direction of rooks, d-for diagonals, the prime (') sign marks the non-attacking propertie, while absence of the prime sign denotes the attacking propertie.

According with the principle of *inclusions and exclusions,* however, we must subtract from this amount of sum all these cases, which we counted more than once due to the position of the queens:

✔ So we counted three times those positions, when all three pieces places on the same diagonal. Let we mark this like $\boldsymbol{\varphi = N(a_1\text{-}a_2\text{-}a_3)}$. Therefore, this should not be included twice, or in other words, we must subtract it twice from the value of $N(t'd)$!
✔ ✔ Moreover we must consider those cases where position of the pieces are such, that for example, the a_1 queen attacks a_2, the a_2 attacks a_3 , but the a_1 dont attacks the a_3 , because these kind of positions we counted two times, so we must subtract this amount once from the value of $N(t'\,d)$. Let we mark these cases with $\boldsymbol{\Delta=N(a_1a_2a_3)}$!
✔ ✔ ✔ Observe yet, that, due to invariance of the referenced labels of pieces, the value of $N(a_1\text{-}a_2)$ is equal with $N(a_2\text{-}a_3)$, thus $\boldsymbol{\sigma = \delta}$!

Therefore respecting the above statements and with appropriate substitutions, we can set up the following logical equation:

$N(t'd)=[2\,N(a_1\text{-}a_2) + N(a_1\text{-}a_3)] - [2\,N(a_1\text{-}a_2\text{-}a_3) + N(a_1a_2a_3)]$, or symbolically

$$\boldsymbol{N(t'\,d)=(2\sigma+\lambda)-(2\varphi+\Delta)} \tag{4}$$

For confirmation of our idea, let we see values of $N(t'd)$ and its components given above for a few number of n -s in the following table, where the adequate values was determined empirically with the so called *„backtrack”* method:

n	***N(t' d)=***	σ	+λ	+ δ	***-2 φ***	**-Δ**
3	6	4	2	4	-4	0
4	72	44	24	44	-24	-16
5	396	210	120	210	-80	-64
6	1376	680	400	680	-200	-184

(Applying the certain but not so easy *„backtrack”* method require cca. 6 hours of hard working to fill out only the last row, even we investigate only the half of the whole table, what is here possible indeed in the case of the n=6 simmetrical table!)

Now, go forth to numerical determination of the particular component, which we also gain using the *„backtrack”* method, by generalizing their values from array of numbers into a mathematical formula!

In the whole course of our work we used the below presented modified tabular form of the *Pascal triangle*, where any value of the m –th row can be interpreted as the m -th class sum of the first n –th natural number array. Thus, expl. ΣΣΣ(5)=70

n	1	2	3	4	5	6	7	8	9	10	…
Σn	1	3	6	10	**15**	21	28	36	45	55	…
ΣΣn	1	4	10	20	**35**	56	84	120	165	220	…
ΣΣΣn	**1**	**5**	**15**	**35**	**70**	**126**	**210**	**330**	**495**	**715**	**…**
ΣΣΣΣn	1	6	21	56	**126**	252	462	792	1287	2002	…
…	…	…	…	…	…	…	…	…	…	…	…

And here we emphasize the general identity which will be important only at a later stage

$$\sum_{1}^{m}(n) = \binom{n+m}{m+1}$$

Because with him, any relationship derived from the above table can be converted into a binomial form. For example:

1: $\Sigma\Sigma\Sigma(5) = \binom{5+3}{3+1} = \binom{8}{4} = 70$

2: $\Sigma\Sigma\Sigma(n\text{-}1) = \binom{n\text{-}1+3}{3+1} = \binom{n+2}{4}$

3: $\Sigma\Sigma(n\text{-}3)] = \binom{n\text{-}3+2}{2+1} = \binom{n-1}{3}$

The other two interesting identity (which will, however, be utilized only in the APPENDIX-II "Closing of the open binomial array"), does not arise directly from the Pascal's table, but are equally well illustrated by the following two tables composed in a similar way:

$\Sigma n^2 = \Sigma n + 2\,\Sigma\Sigma(n\text{-}1)$

$\Sigma n^3 = (\Sigma n)^2$

n	1	2	3	4	5	6	7	8
n^3	1	8	27	64	125	216	343	512
Σn^3	1	9	36	100	225	441	784 …	
$(\Sigma n)^2$	1	9	36	100	225	441	784 …	

n	1	2	3	4	5	6	7	8
n^2	1	4	9	16	25	36	49	64
Σn^2	1	5	14	30	55	91	140 …	
$\Sigma n+2*\Sigma\Sigma(n-1)$	1	5	14	30	55	91	140 …	

Our basic idea involve the conjecture, that the empirical-heuristic values gained by the *„backtrack"* method can be always expressed with some functional relationship of the binomial coeffitients represented by Pascal triangle, depends only on the size of the chess board, and our direct aim is achieving them in intuitive way! So let's go forth gradually.

2. The development process

2.1. Determining the $\sigma = N(a_1\text{-}a_2)$

Here, the following relationship could be established

3	2	2Σ(n-2)			
4	4	2Σ(n-3)	3Σ(n-2)		
5	6	2Σ(n-4)	3Σ(n-3)	4Σ(n-2)	
6	8	2Σ(n-5)	3Σ(n-4)	4Σ(n-3)	5Σ(n-2)
…	…	…	…	…	…
n	**2(*n*-2) [(*n*-1)Σ(*n*-2) + (*n*-2)Σ(*n*-3) + (*n*-3)Σ(*n*-4) + ...]**				

It should also be notice, that with extracting 2(*n*-2) from the sum of items in the infinit open array, the rest, remained in brackets can be clearly and directly expressed with *sum* (!) of 2ΣΣΣ(n-2) and ΣΣΣ(n-3), as shown in the table below:

n	drown out 2(n-2)* =	σ =	2ΣΣΣ(n-2) +	ΣΣΣ(n-3)
3	**2**	2	2	0
4	**2 9**	11	10	1
5	**2 9 24**	35	30	5
6	**2 9 24 50**	85	70	15
7	**2 9 24 50 90**	175	140	35

So we could written in closed form as:

$$\sigma = 2(n\text{-}2)\ [2\Sigma\Sigma\Sigma(n\text{-}2) + \Sigma\Sigma\Sigma(n\text{-}3)] \quad (5)$$

2.2. Determining the $\lambda = N(a_1\text{-}a_3)$

The first -and the third queen attacks shows a completely different structure:

n					2(n-2) drown out
3	2(n-2)x1				1
4	4(n-2)x1	2(n-2)x2			2x1+2
	4(n-2)x1				2x1
5	6(n-2)x1	4(n-2)x2	2(n-2)x3		3x1+2x2+3
	6(n-2)x1	4(n-2)x2			3x1+2x2
	6(n-2)x1				3x1
6	8(n-2)x1	6(n-2)x2	4(n-2)x3	2(n-2)x4	4x1+3x2+2x3+1x4
	8(n-2)x1	6(n-2)x2	4(n-2)x3		4x1+3x2+2x3
	8(n-2)x1	6(n-2)x2			4x1+3x2
	8(n-2)x1				4x1

But extracting 2(*n*-2) also leads us to result, the solution is now given by *difference* of the values ΣΣΣΣ(*n*-2) and ΣΣΣΣ(*n*-4) !

n		λ =	ΣΣΣΣ(n-2)	–ΣΣΣΣ(n-4)
3	**1**	1	1	0
4	**4 2**	6	6	0
5	**10 7 3**	20	21	-1
6	**16 18 12 4**	50	56	-6

$$\lambda = 2(n\text{-}2)\ [\Sigma\Sigma\Sigma\Sigma(n\text{-}2) - \Sigma\Sigma\Sigma\Sigma(n\text{-}4)] \quad (6)$$

2.3. Determining the $\varphi = N(a_1\text{-}a_2\text{-}a_3)$

n		
3	2(ΣΣ(1) +2ΣΣ(0)) =	2(1)
4	2(ΣΣ(2) +2ΣΣ(1) + 2ΣΣ(0)) =	2(4+2*1)
5	2(ΣΣ(3) +2ΣΣ(2) + 2ΣΣ(1) +2ΣΣ(0)) =	2(10+2*4+2*1)
6	2(ΣΣ(4) +2ΣΣ(3) + 2ΣΣ(2) +2ΣΣ(1) +ΣΣ(0)) =	2(20+2*10+2*4+2*1)
7	2(ΣΣ(5) +2ΣΣ(4) + 2ΣΣ(3) +2ΣΣ(2) +ΣΣ(1) +ΣΣ(0)) =	2(35+2*20+2*10+2*4+2*1)

n		φ =	2*	[ΣΣ(n-2)	+2ΣΣ(n-3)	+2ΣΣ(n-4)	+2ΣΣ(n-5)	+2ΣΣ(n-6)]
3	**2**	2	2	(1)				
4	**8 4**	12	2	(4	+2 * 1)			
5	**20 16 4**	40	2	(10	+2 * 4	+2 * 1)		
6	**40 40 16 4**	100	2	(20	+2 * 10	+2 * 4	+2 * 1)	
7	**70 80 40 16 4**	210	2	(35	+2 * 20	+2 * 10	+2 * 4	+2 * 1)

φ =2[ΣΣ(*n*-2) + [2ΣΣ(*n*-3) + 2ΣΣ(*n*-4) +2ΣΣ(*n*-5) +2ΣΣ(*n*-6) + ...]]

Since [2ΣΣ(*n*-3) + 2ΣΣ(*n*-4) +2ΣΣ(*n*-5) +...] = 2ΣΣΣ(n-3)
therefore we can write that

φ =2[ΣΣ(n-2) + 2ΣΣΣ(n-3)] (7)

2.4. Determining the $\Delta = N(a_1a_2a_3)$

The situation here is slightly more difficult. Therefore we confined at first to counting only the basic positions and let we marked them temporarily with $\Psi(n)$:

n				$\Psi(n)$
4	1 + 1			2
5	1 + 2 + 2 1 + 1 + 1			8
6	1 + 2 + 3 + 3 1 + 2 + 2 + 2 1 + 1 + 1 + 1	1 + 1 + 1		23
7	1 + 2 + 3 + 4 + 4 1 + 2 + 3 + 3 + 3 1 + 2 + 2 + 2 + 2 1 + 1 + 1 + 1 + 1	1 + 2 + 2 + 2 1 + 1 + 1 + 1		51
8	1 + 2 + 3 + 4 + 5 + 5 1 +2 + 3 + 4 + 4 + 4 1 + 2 + 3 + 3 + 3 +3 1 + 2 + 2 + 2 + 2 +2 1 + 1 + 1 + 1 + 1 +1	1 + 2 + 3 + 3 + 3 1 + 2 + 2 + 2 + 2 1 + 1 + 1 + 1 + 1	1 + 1 + 1 + 1	100

The table shows a regularity where values of $\Psi(n)$ are given by sums of two well distinguished numeric arrays, mark them namely with $G(n)$ and $F(n)$.

Column	G(n) + F(n)	G(n) + F(n)	G(n) + F(n)
I	1(n-3) + Σ(n-3)	2(n-5) + Σ(n-5)	1(n-3) + Σ(n-3)
II	2(n-4) + Σ(n-4)	3(n-6) + Σ(n-6)	2(n-4) + Σ(n-4)
III	3(n-5) + Σ(n-5)	4(n-7) + Σ(n-7)	3(n-5) + Σ(n-5)
IV	4(n-6) + Σ(n-6)	5(n-8) + Σ(n-8)	4(n-6) + Σ(n-6)

From this, determining the $G(n)$:

I column: 1(n-3)+2(n-4)+3(n-5)+4(n-6)+ ..... = ΣΣ(n-3)
II column: 2(n-5)+3(n-6)+4(n-7)+5(n-8)+ ..... = ΣΣ(n-4) -1(n-4)
III column: 3(n-7)+4(n-8)+5(n-9)+6(n-10)+ ..... = ΣΣ(n-5) -1(n-5)-2(n-6)
IV column: 4(n-9)+5(n-10)+6(n-11)+7(n-12)+ ..... = ΣΣ(n-6) -1(n-6)-2(n-7)-3(n-8)

From the above, $G(n)=\sum_{i=1}^{n} i\, G(n)$

$$\mathbf{G(n)=\Sigma\Sigma\Sigma(n\text{-}3)\ \text{-}[1\Sigma(n\text{-}4)+2\Sigma(n\text{-}6)+3\Sigma(n\text{-}8)+4\Sigma(n\text{-}10)+\dots]} \qquad (8)$$

Definition of the $F(n)$ can be read directly from the basic regularity, because
Σ(n-3) + Σ(n-4) +Σ(n-5) + ... =ΣΣ(n-3) , and
Σ(n-5) + Σ(n-6) +Σ(n-7) + ... =ΣΣ(n-5) , and
Σ(n-7) + Σ(n-8) +Σ(n-9) + ... =ΣΣ(n-7) , etc., therefore

$$\mathbf{F(n)=\Sigma\Sigma(n\text{-}3)+\Sigma\Sigma(n\text{-}5)+\Sigma\Sigma(n\text{-}7)+\Sigma\Sigma(n\text{-}9)+\dots} \qquad (9)$$

Now we can write that:

$$\mathbf{\Psi(n)=F(n)+G(n)}$$

✔ ✔ ✔ ✔ Since 3 Queens under given conditions where no one can be symmetric to any diagonal or axis, therefore each basic position of $\Psi(n)$ can be transformed 8 –times such a way (with reflection and rotation) that they provide a completely new solution! Therefore,

$$\mathbf{\Delta=8\ \Psi(n)} \qquad (10)$$

2.5. Assembling the formula

Now there is given each element of the expression $\mathbf{N(t'\,d)=(2\sigma+\lambda)-(2\varphi+\Delta)}$ outlined above in (4) and used as starting point, so it can be written:

$$\mathbf{N(t'\,d)=\Big[2*2(n\text{-}2)\big[2\Sigma\Sigma\Sigma(n\text{-}2)+\Sigma\Sigma\Sigma(n\text{-}3)\big]+2(n\text{-}2)\big[\Sigma\Sigma\Sigma\Sigma(n\text{-}2)-\Sigma\Sigma\Sigma\Sigma(n\text{-}4)\big]\Big]}$$
$$\mathbf{-\Big[2*2\big[\Sigma\Sigma(n\text{-}2)+2\Sigma\Sigma\Sigma(n\text{-}3)\big]+8\Psi(n)\Big]} \qquad (11)$$

Taking into account the

$$\sum_{1}^{m}(n)=\binom{\boldsymbol{n+m}}{\boldsymbol{m+1}} \quad \text{identity,} \qquad (12)$$

the binomial form of the (11) result above is:

$$N(t'\,d)= \Big[2*2(n\text{-}2)\Big[2\binom{n+1}{4}+\binom{n}{4}\Big] +2(n\text{-}2)\Big[\binom{n+2}{5}\text{-}\binom{n}{5}\Big]\Big]$$

$$\text{-}\Big[2*2\Big[\binom{n}{3}+2\binom{n}{4}\Big] +8\,\Psi(n)\Big] \qquad (13)$$

After combining and simplifying the binomial terms, we find that

$$N(t'd) = 2\binom{n}{3}\left[2n^2 - 6n +3\right] -8\,\Psi(n) \qquad (14)$$

Since $N(t'd') = N(t') - N(t'd)$ according to the (1) basic relation, and $N(t')=\binom{n}{3}^2 3!$ by the expression (2), we thereby finally collected the formula for solving the 3-Queens problem! We can write now the total value of possible combinations as $P_3 = N(t'd')$, or

$$P_3 = \binom{n}{3}^2 3! -2\binom{n}{3}\left[2n^2 - 6n +3\right] +8\,\Psi(n) \qquad (15)$$

where $\Psi(n) =F(n) + G(n)$ or with the binomial forms of (8) and (9)

$$F(n) =\binom{n-1}{3} + \binom{n-3}{3} + \binom{n-5}{3} + \binom{n-7}{3} + \ldots\ldots$$

$$G(n) =\binom{n}{4} - \left[\, 1\binom{n-3}{2} + 2\binom{n-5}{2} + 3\binom{n-7}{2} + \ldots..\right]$$

Combining the arrays $F(n)$ and $G(n)$ we get:

$$\Psi(n) = \binom{n}{4} + \left[1\binom{n-2}{2} + 2\binom{n-4}{2} + 3\binom{n-6}{2} + \;\ldots.\right] \qquad (16)$$

Let we add the firs item of (16) to formula in (15) and mark the rest of (16) with $Q(n)$, than we find that

$$P_3 = \binom{n}{3}^2\; 3! -2\binom{n}{3}\left[2n^2 - 7n + 6\right] + 8\,Q(n) \quad \text{where is} \qquad (17)$$

$$Q(n) =1\binom{n-2}{2} + 2\binom{n-4}{2} + 3\binom{n-6}{2} + \ldots \quad \text{still an open infinite array!} \qquad (18)$$

Now it's worth to deal with closing the expression (18), (which is done in the APPENDIX-II), thus it gives

$$Q(n) = \frac{1}{192}\left[3(-1)^n\;(2n-1)+2n^4 -4n^3 -8n^2 +10n+3\right] \qquad (19)$$

Substituting the latter (19) result to the (17) formula above, we finally find that

$$\boxed{P_3 = \binom{n}{3}^2\; 3! -2\binom{n}{3}\left[2n^2 -7n +6\right] + \frac{1}{24}\left[3(-1)^n\;(2n -1) +2n^4 -4n^3 -8n^2 +10n +3\right]} \qquad (20)$$

which is nothing else than the final and complete solution of the 3 -Queens problem!

3. Notes

Of interesting is to be mentioned, that the above formula can be rearranged in such a form, which is very reminiscent of the formula for solving the 2 Queens problem, since:

$$\boldsymbol{P_3} = \frac{1}{24}\left[k(-1)^{kn}(2n\text{-}1) + 4n^6 - 40n^5 + 158n^4 - 300n^3 + 264n^2 - 86n + 3\right] \text{ and}$$
$$\boldsymbol{P_2} = \frac{1}{6}\left[k(-1)^{kn} + 3n^4 - 10n^3 + 9n^2 - 2n - 2\right]$$

We don't continue the possible analogies, since the goal was here the formula for solving the three queens problem. Otherwise, the method is very labor intensive, the above results needs three-month intensive effort, therefore the task of resolving the problem in higher case or any number of N-Queens, I leave to others... (*)

Returning to the (1) basic relation, we can now instantly fill out any row of the table below for illustration, so for example, in the case of n = 80 the $\boldsymbol{P_3}$ exact number of all possible combinations is 38,492,656,800.

n	N(t')	-N(t'd)	P_3=N(f'd')
3	6	6	0
4	96	72	24
5	600	396	204
6	2,400	1,376	1,024
7	7,350	3,722	3,628
8	18,816	8,496	10,320
9	42,336	17,240	25,096
10	86,400	32,000	54,400
11	163,350	55,470	107,880
12	290,400	91,000	199,400
13	490,776	142,756	348,020
14	794,976	215,712	579,264
15	1,242,150	315,826	926,324
16	1,881,600	450,016	1,431,584
17	2,774,400	626,352	2,148,048
18	3,995,136	854,016	3,141,120
19	5,633,766	1,143,510	4,490,256
20	7,797,600	1,506,600	6,291,000
50	2,304,960,000	182,016,000	2,122,944,000
60	7,026,050,400	463,391,000	6,562,659,400
80	40,501,593,600	2,008,936,800	38,492,656,800

EPILOGUE

(*) Note that the above statement concerns the complex situation at 38 years ago, in conditions when there was no other methods available than the "backtrack" method, with manually registering and counting, which would be an enormous time consumptive for any number of ***k*** greater than 3! But , on the other hand, nowadays, using services of fast computers, we could obtain very quickly all the required numeric values even for a much higher number of ***k***, therefore, the presented method actually appears as very feasible way to go forward, and seems a good tool to finding out the final numerical solution for the generalized ***nxn*** chess board!

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